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ON THE HYERS-ULAM STABILITY OF AN ADDITIVE FUNCTIONAL INEQUALITY

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x-y)+f(y-z)+f(z)\| \leq \|f(x)\|$$

in Banach spaces.

1. Introduction and preliminaries

In 1940, Ulam [6] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows: *Let $(\mathcal{G}, \circ)$ be a group and let $(\mathcal{H}, \star, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta = \delta(\varepsilon) > 0$ such that if a mapping $f : \mathcal{G} \to \mathcal{H}$ satisfies the inequality*

$$d\big(f(x \circ y), f(x) \star f(y)\big) < \delta$$

for all $x, y \in \mathcal{G}$, then a homomorphism $F : \mathcal{G} \to \mathcal{H}$ exits with

$$d\big(f(x), F(x)\big) < \varepsilon$$

for all $x \in \mathcal{G}$?

In 1941, Hyers [2] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces as follows: *If $\delta > 0$ and if $f : \mathcal{E} \to \mathcal{F}$ is a mapping between Banach spaces $\mathcal{E}$ and $\mathcal{F}$ satisfying*

$$\big\|f(x+y) - f(x) - f(y)\big\| \leq \delta$$

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for all $x, y \in \mathcal{E}$, *then there is a unique additive mapping* $A : \mathcal{E} \to \mathcal{F}$ *such that*

$$\big\| f(x) - A(x) \big\| \leq \delta$$

for all $x, y \in \mathcal{E}$.

We will recall a fundamental result in fixed point theory for explicit later use.

THEOREM 1.1. (The alternative of fixed point) [1, 5]
Suppose we are given a complete generalized metric space $(\mathcal{X}, d)$ *and a strictly contractive mapping* $\Lambda : \mathcal{X} \to \mathcal{X}$, *with the Lipschitz constant* L. *Then, for each given element* $x \in \mathcal{X}$, *either*

$$d(\Lambda^n x, \Lambda^{n+1} x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*

(a) $d(\Lambda^n x, \Lambda^{n+1}) < \infty$ *for all* $n \geq n_0$;
(b) *The sequence* $(\Lambda^n x)$ *is convergent to a fixed point* y^* *of* Λ;
(c) y^* *is the unique fixed point of* Λ *in the set* $Y = \{y \in X | d(\Lambda^{n_0}, y) < \infty\}$;
(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, \Lambda y)$ *for all* $y \in Y$.

2. Hyers-Ulam stability in Banach spaces

Throughout this paper, let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a Banach space. In 2007, Park, Cho and Han [4] proved the Hyers-Ulam stability of the additive functional inequality

$$\big\| f(x) + f(y) + f(z) \big\| \leq \big\| f(x + y + z) \big\|$$

in Banach spaces. In 2011, Lee, Park and Shin [3] prove the Hyers-Ulam stability of the additive functional inequality

$$\big\| f(2x) + f(2y) + 2f(z) \big\| \leq \big\| 2f(x + y + z) \big\|$$

in Banach spaces.

In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\| f(x - y) + f(y - z) + f(z) \| \leq \| f(x) \|$$

in Banach spaces.

LEMMA 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. Then it is additive if and only if it satisfies*

$$\|f(x-y)+f(y-z)+f(z)\| \le \|f(x)\| \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$.

Proof. If f is additive, then clearly

$$\big\|f(x-y)+f(y-z)+f(z)\big\| = \big\|f(x)\big\|$$

for all $x, y, z \in \mathcal{X}$.

Assume that f satisfies (2.1). Letting $x = y = z = 0$ in (2.1), we gain $\big\|3f(0)\big\| \le \big\|f(0)\big\|$ and so $f(0) = 0$. Putting $x = z = 0$ in (2.1), we get

$$\big\|f(-y)+f(y)\big\| \le \big\|f(0)\big\| = 0$$

and so $f(-y) = -f(y)$ for all $y \in \mathcal{X}$. Letting $x = 0$ and replacing z by $-z$ in (2.1), we have

$$\big\|f(y+z)+f(-y)+f(-z)\big\| \le \big\|f(0)\big\| = 0$$

for all $y, z \in \mathcal{X}$. Thus we obtain

$$f(y+z) = f(y)+f(z)$$

for all $y, z \in \mathcal{X}$. □

THEOREM 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying*

$$\|f(x-y)+f(y-z)+f(z)\| \le \|f(x)\| + \varphi(x, y, z) \tag{2.2}$$

and

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\big((-2)^j x, (-2)^j y, (-2)^j z\big) < \infty \tag{2.3}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - A(x)\big\| \le \frac{1}{2}\widetilde{\varphi}(0, -x, x) \tag{2.4}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $0, -(-2)^n x, (-2)^n x$, respectively, and dividing by 2^{n+1} in (2.2), since $f(0) = 0$, we get

$$\left\|\frac{f\big((-2)^{n+1}x\big)}{(-2)^{n+1}} - \frac{f\big((-2)^n x\big)}{(-2)^n}\right\| \le \frac{1}{2^{n+1}}\varphi\big(0, -(-2)^n x, (-2)^n x\big)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. From the above inequality, we have

$$\begin{aligned} \left\| \frac{f\big((-2)^n x\big)}{(-2)^n} - \frac{f\big((-2)^m x\big)}{(-2)^m} \right\| &\le \sum_{j=m}^{n-1} \left\| \frac{f\big((-2)^{j+1} x\big)}{(-2)^{j+1}} - \frac{f\big((-2)^j x\big)}{(-2)^j} \right\| \\ &\le \sum_{j=m}^{n-1} \frac{1}{2^{j+1}} \varphi\big(0, -(-2)^j x, (-2)^j x\big) \end{aligned} \tag{2.5}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. By the condition (2.3), the sequence $\left\{ \frac{f((-2)^n x)}{(-2)^n} \right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{ \frac{f((-2)^n x)}{(-2)^n} \right\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f\big((-2)^n x\big)}{(-2)^n}$$

for all $x \in \mathcal{X}$. Taking $m = 0$ and letting n tend to ∞ in (2.5), we have the inequality (2.4).

Replacing x, y, z by $(-2)^n x, (-2)^n y, (-2)^n z$, respectively, and dividing by 2^n in (2.2), we obtain

$$\begin{aligned} &\left\| \frac{f\big((-2)^n (x-y)\big)}{(-2)^n} + \frac{f\big((-2)^n (y-z)\big)}{(-2)^n} + \frac{f\big((-2)^n (z)\big)}{(-2)^n} \right\| \\ &\le \left\| \frac{2f\big((-2)^n (x)\big)}{(-2)^n} \right\| + \frac{1}{2^n} \varphi\big((-2)^n x, (-2)^n y, (-2)^n z\big) \end{aligned}$$

for all $x, y, z \in \mathcal{X}$ and all nonnegative integers n. Since (2.3) gives that

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi\big((-2)^n x, (-2)^n y, (-2)^n z\big) = 0$$

for all $x, y, z \in \mathcal{X}$, letting n tend to ∞ in the above inequality, we see that A satisfies the inequality (2.1) and so it is additive by Lemma 2.1.

Let $A' : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (2.4). Since both A and A' are additive, we have

$$\begin{aligned}
&\big\|A(x) - A'(x)\big\| \\
&= \frac{1}{2^n}\big\|A\big((-2)^n x\big) - A'\big((-2)^n x\big)\big\| \\
&\le \frac{1}{2^n}\big(\big\|A\big((-2)^n x\big) - f\big((-2)^n x\big)\big\| + \big\|f\big((-2)^n x\big) - A'\big((-2)^n x\big)\big\|\big) \\
&\le \frac{1}{2^n}\widetilde{\varphi}\big(0, -(-2)^n x, (-2)^n x\big) \\
&= \sum_{j=n}^{\infty} \frac{1}{2^j}\varphi\big(0, -(-2)^j x, (-2)^j x\big)
\end{aligned}$$

which goes to zero as $n \to \infty$ for all $x \in \mathcal{X}$ by (2.3). Therefore, A is a unique additive mapping satisfying (2.4), as desired. $\square$

Corollary 2.3. *Let $\theta \in [0,\infty)$ and $p \in [0,1)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x-y) + f(y-z) + f(z)\| \le \|f(x)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.6}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p \tag{2.7}$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 2.2, take $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in \mathcal{X}$. Then we have the desired result. $\square$

Theorem 2.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0,\infty)$ satisfying (2.2) and*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{(-2)^j}, \frac{y}{(-2)^j}, \frac{z}{(-2)^j}\right) < \infty \tag{2.8}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - A(x)\big\| \le \frac{1}{2}\widetilde{\varphi}(0, -x, x) \tag{2.9}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $0, \frac{-x}{(-2)^n}, \frac{x}{(-2)^n}$, respectively, and multiplying by 2^{n-1} in (2.2), since $f(0)=0$, we have

$$\left\|(-2)^n f\left(\frac{x}{(-2)^n}\right)-(-2)^{n-1} f\left(\frac{x}{(-2)^{n-1}}\right)\right\| \leq 2^{n-1}\varphi\left(0, \frac{-x}{(-2)^n}, \frac{x}{(-2)^n}\right)$$

for all $x \in \mathcal{X}$ and all $n \in \mathbb{N}$. From the above inequality, we get

$$\begin{aligned}
(2.10)\qquad & \left\|(-2)^n f\left(\frac{x}{(-2)^n}\right) - (-2)^m f\left(\frac{x}{(-2)^m}\right)\right\| \\
& \leq \sum_{j=m+1}^{n} \left\|(-2)^j f\left(\frac{x}{(-2)^j}\right) - (-2)^{j-1} f\left(\frac{x}{(-2)^{j-1}}\right)\right\| \\
& \leq \sum_{j=m+1}^{n} 2^{j-1}\varphi\left(0, \frac{-x}{(-2)^j}, \frac{x}{(-2)^j}\right)
\end{aligned}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. From (2.8), the sequence $\left\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in \mathcal{X}$. To prove that A satisfies (2.9), putting $m = 0$ and letting $n \to \infty$ in (2.10), we have

$$\|f(x) - A(x)\| \leq \sum_{j=1}^{\infty} 2^{j-1}\varphi\left(0, \frac{-x}{(-2)^j}, \frac{x}{(-2)^j}\right) = \frac{1}{2}\widetilde{\varphi}(0, -x, x)$$

for all $x \in \mathcal{X}$.

Replacing x, y, z by $\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}$, respectively, and multiplying by 2^n in (2.2), we obtain

$$\begin{aligned}
& \left\|(-2)^n f\left(\frac{x-y}{(-2)^n}\right) + (-2)^n f\left(\frac{y-z}{(-2)^n}\right) + (-2)^n f\left(\frac{z}{(-2)^n}\right)\right\| \\
& \leq \left\|(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\| + 2^n\varphi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\right)
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ and all nonnegative integers n. Since (2.8) gives that

$$\lim_{n\to\infty} 2^n\varphi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\right) = 0$$

for all $x, y, z \in \mathcal{X}$, if we let $n \to \infty$ in the above inequality, then we have

$$\big\|A(x-y) + A(y-z) + A(z)\big\| \le \big\|A(x)\big\|$$

for all $x, y, z \in \mathcal{X}$. By Lemma 2.1, the mapping A is additive. The rest of the proof is similar to the corresponding part of the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $p > 1$ and θ be non-negative real numbers and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x-y) + f(y-z) + f(z)\| \le \|f(x)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{2.11}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^p - 2}\|x\|^p \tag{2.12}$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 2.4, take $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in \mathcal{X}$. Then, we have the desired result. $\square$

3. Hyers-Ulam stability using fixed point methods

Now, using the fixed point method, we investigate the Hyers-Ulam stability of the functional inequality (2.1) in Banach spaces.

Theorem 3.1. *Suppose that an odd mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|f(x-y) + f(y-z) + f(z)\| \le \|f(x)\| + \phi(x, y, z) \tag{3.1}$$

for all $x, y, z \in \mathcal{X}$, where $\phi : \mathcal{X}^3 \to [0, \infty)$ is a function. If there exists $L < 1$ such that

$$\phi(x, y, z) \le \frac{1}{2}L\phi(2x, 2y, 2z) \tag{3.2}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x) - A(x)\| \le \frac{L}{2 - 2L}\phi(0, -x, x) \tag{3.3}$$

for all $x \in \mathcal{X}$.

Proof. Consider a set $S := \{g \,|\, g : \mathcal{X} \to \mathcal{Y}\}$ and introduce a generalized metric d on S as follows:

$$d(g,h) = d_\phi(g,h) := \inf S_\phi(g,h),$$

where

$$S_\phi(g,h) := \{C \in (0,\infty) : \|g(x) - h(x)\| \le C\phi(0,-x,x) \text{ for all } x \in \mathcal{X}\}$$

for all $g,h \in S$. Now we show that (S,d) is complete. Let $\{h_n\}$ be a Cauchy sequence in (S,d). Then, for any $\varepsilon > 0$ there exists an integer $N_\varepsilon > 0$ such that $d(h_m,h_n) < \varepsilon$ for all $m,n \ge N_\varepsilon$. Since $d(h_m,h_n) = \inf S_\phi(h_m,h_n) < \varepsilon$ for all $m,n \ge N_\varepsilon$, there exists $C \in (0,\varepsilon)$ such that

$$\|h_m(x) - h_n(x)\| \le C\phi(0,-x,x) \le \varepsilon\phi(0,-x,x) \tag{3.4}$$

for all $m,n \ge N_\varepsilon$ and all $x \in \mathcal{X}$. So $\{h_n(x)\}$ is a Cauchy sequence in $\mathcal{Y}$ for each $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, $\{h_n(x)\}$ converges for each $x \in \mathcal{X}$. Thus a mapping $h : \mathcal{X} \to \mathcal{Y}$ can be defined by

$$h(x) := \lim_{n\to\infty} h_n(x) \tag{3.5}$$

for all $x \in \mathcal{X}$. Letting $n \to \infty$ in (3.4), we have

$$\begin{aligned} m \ge N_\varepsilon &\Rightarrow \|h_m(x) - h(x)\| \le \varepsilon\phi(0,-x,x) \\ &\Rightarrow \varepsilon \in S_\phi(h_m,h) \\ &\Rightarrow d(h_m,h) = \inf S_\phi(h_m,h) \le \varepsilon \end{aligned}$$

for all $x \in \mathcal{X}$. This means that the Cauchy sequence $\{h_n\}$ converges to h in (S,d). Hence (S,d) is complete.

Define a mapping $\Lambda : S \to S$ by

$$\Lambda h(x) := 2h\Big(\frac{x}{2}\Big) \tag{3.6}$$

for all $x \in \mathcal{X}$. We claim that Λ is strictly contractive on S. For any given $g,h \in S$, let $C_{gh} \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le C_{gh}$. Then

$$\begin{aligned} &d(g,h) \le C_{gh} \\ &\Rightarrow \|g(x) - h(x)\| \le C_{gh}\phi(0,-x,x) \quad \text{for all } x \in \mathcal{X} \\ &\Rightarrow \Big\|2g\Big(\frac{x}{2}\Big) - 2h\Big(\frac{x}{2}\Big)\Big\| \le 2C_{gh}\phi\Big(0,-\frac{x}{2},\frac{x}{2}\Big) \quad \text{for all } x \in \mathcal{X} \\ &\Rightarrow \Big\|2g\Big(\frac{x}{2}\Big) - 2h\Big(\frac{x}{2}\Big)\Big\| \le LC_{gh}\phi(0,-x,x) \quad \text{for all } x \in \mathcal{X}, \end{aligned}$$

that is, $d(\Lambda g, \Lambda h) \le LC_{gh}$. Hence we see that $d(\Lambda g, \Lambda h) \le Ld(g,h)$ for any $g,h \in S$. Therefore Λ is strictly contractive mapping on S with the

Lipschitz constant $L \in (0,1)$. Putting $x = 0$, $y = -x$ and $z = x$ in (3.1), we have

$$\|f(2x) - 2f(x)\| \leq \phi(0, -x, x) \tag{3.7}$$

for all $x \in \mathcal{X}$. It follows from (3.7) that

$$\left\| f(x) - 2f\Big(\frac{x}{2}\Big) \right\| \leq \phi\Big(0, -\frac{x}{2}, \frac{x}{2}\Big) \leq \frac{L}{2}\phi(0, -x, x) \tag{3.8}$$

for all $x \in \mathcal{X}$. Thus $d(f, \Lambda f) \leq \frac{L}{2}$. Therefore, it follows from Theorem 1.1 that the sequence $\{\Lambda^n f\}$ converges to a fixed point A of Λ, i.e.,

$$A : \mathcal{X} \to \mathcal{Y}, \quad A(x) = \lim_{n\to\infty} (\Lambda f)(x) = \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big)$$

and $A(2x) = 2A(x)$ for all $x \in \mathcal{X}$. Also A is the unique fixed point of Λ in the set $S^* = \{g \in S \,|\, d(f, g) < \infty\}$ and

$$d(A, f) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{L}{2-2L},$$

i.e., the inequality (3.3) holds for all $x \in \mathcal{X}$. It follows from the definition of A and (3.1) that

$$\big\|A(x-y) + A(y-z) + A(z)\big\| \leq \big\|A(x)\big\|$$

for all $x, y, z \in \mathcal{X}$. By Lemma 2.1, the mapping $A : \mathcal{X} \to \mathcal{Y}$ is a Cauchy additive mapping. Therefore, there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying (3.3). $\square$

COROLLARY 3.2. *Let $p > 1$ and θ be non-negative real numbers and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x-y) + f(y-z) + f(z)\| \leq \|f(x)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{3.9}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{2^p + 1}{2^p - 2}\theta\|x\|^p \tag{3.10}$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 3.1, take $\phi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in \mathcal{X}$. Then, we can choose $L = 2^{1-p}$ and we have the desired result. $\square$

THEOREM 3.3. *Suppose that an odd mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|f(x-y) + f(y-z) + f(z)\| \leq \|f(x)\| + \phi(x,y,z) \tag{3.11}$$

for all $x, y, z \in \mathcal{X}$, where $\phi : \mathcal{X}^3 \to [0,\infty)$ is a function. If there exists $L < 1$ such that

$$\phi(x,y,z) \leq 2L\phi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big) \tag{3.12}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x) - A(x)\| \leq \frac{1}{2-2L}\phi(0,-x,x) \tag{3.13}$$

for all $x \in \mathcal{X}$.

Proof. Consider the complete generalized metric space (S, d) given in the proof of Theorem 3.1. Now we consider the linear mapping $\Lambda : S \to S$ given by

$$\Lambda h(x) = \frac{1}{2}h(2x)$$

for all $x \in \mathcal{X}$. For any given $g, h \in S$, let $C_{gh} \in [0,\infty]$ be an arbitrary constant with $d(g,h) \leq C_{gh}$. Hence we obtain

$$d(\Lambda g, \Lambda h) \leq Ld(g,h)$$

for all $g, h \in S$. It follows from (3.7) that $d(f, \Lambda f) \leq \frac{1}{2}$. The rest of the proof is similar to the corresponding part of the proof of Theorem 3.1. □

COROLLARY 3.4. *Let $\theta \in [0,\infty)$ and $p \in [0,1)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\|f(x-y) + f(y-z) + f(z)\| \leq \|f(x)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{3.14}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{1+2^p}{2-2^p}\theta\|x\|^p \tag{3.15}$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 3.3, take $\phi(x,y,z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ for all $x, y, z \in \mathcal{X}$. Then we can choose $L = 2^{p-1}$ and we have the desired result. □

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